

On Kleinbock's Diophantine result

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Abstract. We give an elementary proof of a recent metrical Diophantine result by D. Kleinbock related to badly approximable vectors in affine subspaces.

1. Definitions and notation. Let \mathbb{R}^d be a Euclidean space with the coordinates (x_1, \dots, x_d) , let \mathbb{R}^{d+1} be a Euclidean space with the coordinates (x_0, x_1, \dots, x_d) . For $\mathbf{x} \in \mathbb{R}^d$ or $\mathbf{x} \in \mathbb{R}^{d+1}$ we denote by $|\mathbf{x}|$ its sup-norm:

$$|\mathbf{x}| = \max_{1 \leq j \leq d} |x_j| \quad \text{or} \quad |\mathbf{x}| = \max_{0 \leq j \leq d} |x_j|.$$

Consider an affine subspace $A \in \mathbb{R}^d$ and define the affine subspace $\mathcal{A} \in \mathbb{R}^{d+1}$ in the following way:

$$\mathcal{A} = \{\mathbf{x} = (1, x_1, \dots, x_d) : (x_1, \dots, x_d) \in A\}.$$

Let B be another affine subspace such that $B \subset A$. Define

$$\mathcal{B} = \{\mathbf{x} = (1, x_1, \dots, x_d) : (x_1, \dots, x_d) \in B\}, \quad \mathcal{B} \subset \mathcal{A}.$$

Put

$$a = \dim A = \dim \mathcal{A}, \quad b = \dim B = \dim \mathcal{B}.$$

We define *linear* subspaces

$$\mathfrak{A} = \text{span } \mathcal{A}, \quad \mathfrak{B} = \text{span } \mathcal{B}$$

as the smallest linear subspaces in \mathbb{R}^{d+1} containing \mathcal{A} or \mathcal{B} respectively. So

$$\dim \mathfrak{A} = a + 1, \quad \dim \mathfrak{B} = b + 1.$$

Let $\psi(T)$, $T \geq 1$ be a positive valued function decreasing to zero as $T \rightarrow +\infty$. We define an affine subspace B to be ψ -badly approximable if

$$\inf_{\mathbf{x} \in \mathbb{Z}^{d+1} \setminus \{0\}} \left(\frac{1}{\psi(|\mathbf{x}|)} \inf_{\mathbf{y} \in \mathfrak{B}} |\mathbf{x} - \mathbf{y}| \right) > 0. \quad (1)$$

This definition is very convenient for our exposition.

Here we would like to note that in the case when the affine subspace B has zero dimension (and hence $B = \{\mathbf{w}\}$ consists of just one nonzero vector $\mathbf{w} = (w_1, \dots, w_d) \in \mathbb{R}^d$) the definition (1) gives

$$\inf_{\mathbf{x} \in \mathbb{Z}^{d+1} \setminus \{0\}} \left(\frac{1}{\psi(|\mathbf{x}|)} \inf_{t \in \mathbb{R}} |x - t\mathbf{w}^*| \right) > 0 \quad (2)$$

with $\mathbf{w}^* = (1, w_1, \dots, w_d)$. Inequality (2) is equivalent to the condition that there exists a positive constant $\gamma = \gamma(\mathbf{w})$ such that

$$\max_{1 \leq j \leq d} \|w_j x\| \geq \gamma \cdot \psi(|x|), \quad \forall x \in \mathbb{Z} \setminus \{0\} \quad (3)$$

(here $\|\cdot\|$ denotes the distance to the nearest integer). The condition (3) is a usual condition of ψ -badly simultaneously approximable vector. This is an explanation our definition (1).

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In the sequel we consider two positive decreasing functions $\psi(T), \varphi(T)$ under the conditions

$$\varphi(T) \leq \psi(T), \quad \lim_{T \rightarrow +\infty} \psi(T) = 0.$$

For these functions and $R \geq 1$ define

$$\mu_T^{(R)} = \left(\frac{T}{\psi(RT)} \right)^{a-b}, \quad \lambda_T^{(R)} = \left(\frac{\varphi(RT)}{T} \right)^a - \left(\frac{\varphi(R(T+1))}{T+1} \right)^a.$$

2. The result. Now we are ready to formulate the main result of this paper.

Theorem 1. *Let $R \geq 1$. Suppose that $B \subset A$ are affine subspaces and $0 \leq b = \dim B < a = \dim A \leq d$. Let B be a ψ -badly approximable affine subspace. Suppose that $\varphi(T) \leq \psi(T)$ and the series*

$$\sum_{T=1}^{+\infty} \mu_T^{(R)} \lambda_T^{(R)} \quad (4)$$

converges. Then almost all (in the sense of Lebesgue measure) vectors $\mathbf{w} \in A$ such that $|\mathbf{w}| \leq R$ are φ -badly approximable vectors.

We consider a special case $b = 0$ and $\psi(T) = T^{-1/d}$. Then ψ -badly approximable vectors \mathbf{w} are known as simultaneously badly approximable vectors (see [1], Chapter 2). Take

$$\varphi(T) = \psi(T) \cdot (\log T)^{-\Delta}.$$

In the case

$$\Delta > \frac{1}{a}$$

the series (4) converges. So we have the following

Corollary 1. *Consider an affine subspace $A \subset \mathbb{R}^d$. Suppose that there exists a badly approximable vector $\mathbf{w} \in A$. Suppose that $\Delta > \frac{1}{a}$. Then almost all vectors from the subspace A are $\frac{1}{T^{1/d}(\log T)^\Delta}$ -badly approximable vectors.*

Here we should note that in the case $b = 1$ and $\psi(T) = T^{-1/d}$ such a result was obtained recently by Dmitry Kleinbock (see [2], Theorem 4.2) by means of theory of dynamics on homogeneous spaces.

3. Proof of Theorem 1. Given $R \geq 1$ it is enough to prove that almost all vectors $\mathbf{w} = (1, w_1, \dots, w_d) \in \mathcal{A}$ such that $|w_j| \leq R$ are φ -badly approximable in the sense of the definition (2). In the sequel we do not take care on the constants. All constants in the symbols \ll, \asymp may depend on d , subspaces A, B and R .

For a set $\mathfrak{C} \subset \mathbb{R}^{d+1}$ and a point $\mathbf{x} \in \mathbb{R}^{d+1}$ we define the distance $|\mathbf{x}|_{\mathfrak{C}}$ from \mathbf{x} to \mathfrak{C} by

$$|\mathbf{x}|_{\mathfrak{C}} = \inf_{\mathbf{y} \in \mathfrak{C}} |\mathbf{x} - \mathbf{y}|.$$

Consider the set

$$\Omega_T = \{\mathbf{z} \in \mathbb{R}^{d+1} : 0 \leq z_0 \leq T, \max_{1 \leq j \leq d} |z_j| \leq RT, |\mathbf{z}|_{\mathfrak{B}} \leq \gamma \cdot \psi(RT)\}$$

where $\gamma > 0$ is a lower bound for the infimum in (1). As B is a ψ -badly approximable subspace we see that

$$\Omega_T \cap \mathbb{Z}^{d+1} = \{\mathbf{0}\}$$

(we use the definition (1)). Now we observe that any translation of the $1/2$ -dilated set

$$\frac{1}{2} \cdot \Omega_T + \mathbf{c}, \quad \mathbf{c} \in \mathbb{R}^{d+1} \quad (5)$$

consists of not more than one integer point. Indeed if two different integer points \mathbf{x}, \mathbf{y} belong to the same set of the form (5) then $\mathbf{0} \neq \mathbf{x} - \mathbf{y} \in \Omega_T$. This is not possible.

Consider the set

$$\Pi_T = \{\mathbf{z} \in \mathbb{R}^{d+1} : 0 \leq z_0 \leq T, \max_{1 \leq j \leq d} |z_j| \leq RT, |\mathbf{z}|_{\mathfrak{A}} \leq \varphi(RT)\}.$$

As $\varphi(T) \leq \psi(T)$ we see that this set can be covered by not more than $\nu_T \ll \mu_T^{(R)}$ different sets of the form (5). So we deduce a conclusion about an upper bound for the number of integer points in Π_T :

$$\#(\Pi_T \cap \mathbb{Z}^{d+1}) \ll \mu_T^{(R)}. \quad (6)$$

For an integer $T \geq 1$ we consider the set

$$\mathbb{Z}_T = \{\mathbf{z} = (z_0, z_1, \dots, z_d) \in \mathbb{Z}^{d+1} : z_0 = T, \max_{1 \leq j \leq d} |z_j| \leq RT, |\mathbf{z}|_{\mathfrak{A}} \leq \varphi(RT)\}$$

of the cardinality

$$\zeta_T^{(R)} = \#\mathbb{Z}_T, \quad 0 \leq \zeta_T^{(R)} \ll T^a. \quad (7)$$

For $\rho > 0$ and $\mathbf{z} \in \mathbb{R}^{d+1}$ put

$$\mathfrak{U}_\rho(\mathbf{z}) = \{\mathbf{y} \in \mathbb{R}^{d+1} : |\mathbf{z} - \mathbf{y}| \leq \rho\}.$$

Consider the set

$$\mathfrak{U}_T = \bigcup_{\mathbf{z}} \mathfrak{U}_{\varphi(RT)}(\mathbf{z})$$

where the union is taken over all integer points \mathbf{z} such that

$$z_0 = T, \quad \max_{1 \leq j \leq d} |z_j| \leq RT, \quad \mathfrak{U}_{\varphi(RT)}(\mathbf{z}) \cap \mathfrak{A} \neq \emptyset.$$

Clearly

$$\mathfrak{U}_T = \bigcup_{\mathbf{z} \in \mathbb{Z}_T} \mathfrak{U}_{\varphi(RT)}(\mathbf{z})$$

Consider the cone

$$\mathfrak{G}_T = \{\mathbf{x} \in \mathbb{R}^{d+1} : \mathbf{x} = t \cdot \mathbf{y}, t \in \mathbb{R}, \mathbf{y} \in \mathfrak{U}_T\}$$

and the projection

$$\mathcal{U}_T = \mathcal{A} \cap \mathfrak{G}_T.$$

Consider the series

$$\sum_{T=1}^{+\infty} \text{mes}_a \mathcal{U}_T \quad (8)$$

where mes_a stands for the a -dimensional Lebesgue measure. By the Borel-Cantelli lemma arguments Theorem 1 follows from the convergence of the series (8).

Note that the set \mathfrak{U}_T is a union of not more than $\zeta_T^{(R)}$ balls (in sup-norm) of the radius $\varphi(RT)$. Hence the set $\mathcal{U}_T \subset \mathcal{A}$ can be covered by not more than $\zeta_T^{(R)}$ balls of the radius $\varphi(RT)/T$. So in order to prove the convergence of the series (8) one can establish the convergence of the series

$$\sum_{T=1}^{+\infty} \zeta_T^{(R)} \cdot \left(\frac{\varphi(RT)}{T} \right)^a. \quad (9)$$

It follows from (6) that

$$\sum_{j=1}^T \zeta_j^{(R)} \ll \mu_T^{(R)}.$$

Now the convergence of the series (9) follows from the convergence of the series (4) by partial summation as from (7) we see that

$$\zeta_T^{(R)} \cdot \left(\frac{\varphi(R(T+1))}{T+1} \right)^a \ll (\varphi(R(T+1)))^a \rightarrow 0, \quad T \rightarrow +\infty.$$

Theorem 1 is proved.

References

- [1] W.M. Schmidt, Diophantine Approximations, Lect. Not. Math., 785 (1980)
- [2] D. Kleinbock, Extremal supspaces and their submanifolds, GAFA 13:2 (2003), 437 - 466.